The answer is 47

Background

In mathematical number theory, a *divisor function* is an arithmetic function related to the divisors of integers.

The *sum of positive divisors function* $\sigma_x(n)$, for a real or complex number x, is defined as the sum of the xth powers of the positive divisors of n. It can be expressed in sigma notation as

$$\sigma_x(n) = \sum_{d|n} d^x$$

where $d|n$ is shorthand for "d divides n".

When x is 0, the function is referred to as the *number-of-divisors function* or simply the *divisor function*. The notations $d(n)$, $\nu(n)$ and $\tau(n)$ are also used to denote $\sigma_0(n)$. $\tau(n)$ is sometimes called the *τ function* (the *tau function*). It counts the *number of (positive) divisors d of n*. For n=1, 2, 3, ..., the first few values are 1, 2, 2, 3, 2, 4, 2, 4, 3, 4, 2, 6, 2, 4, 4, 5, 2, 6, 2, 6, 4, 4, 2, 8, 3, … (sequence A000005 in OEIS).

When x is 1, the function is called the *sigma function* or *sum-of-divisors function*, and the subscript is often omitted, so $\sigma(n)$ is equivalent to $\sigma_1(n)$.

By analogy with the *sum of positive divisors function* above, let

$$\pi(n) = \prod_{d|n} d$$

denote the *product* of the (positive) divisors d of n (including n itself). For n=1, 2, 3, ..., the first few values are 1, 2, 3, 8, 5, 36, 7, 64, 27, 100, 11, 1728, 13, 196, ... (sequence A007955).

The *divisor product* (i.e. the product of all *positive* divisors) satisfies the identity

$$\pi(n) = \sqrt{n^{\sigma_0(n)}}$$

and a proof of that can be found here: http://planetmath.org/sites/default/files/texpdf/41782.pdf.

A sequence that is always positive when *n* is greater than 47

A lot of mathematical number theory concerns the *positive* divisors of integers.

Let us have a look at the number of *all* divisors (both positive and negative). For n=1, 2, 3, ..., the first few values are 2, 4, 4, 6, 4, 8, 4, 8, 6, 8, 4, 12, 4, 8, 8, 10, 4, 12, 4, 12, 8, 8, 4, 16, 6, … (sequence A062011).

Let us now look at the *product of all divisors*, both positive and negative. The first few values of this sequence is -1, 4, 9, -64, 25, 1296, 49, 4096, -729, … (sequence A217854).

This *extended* divisor product (i.e. the product of *all* divisors), which I denote $\rho(n)$, satisfies the identity

$$\rho(n) = (-n)^{\tau(n)}$$

where

$$\tau(n) = \sigma_0(n) = \sum_{d|n} d^0$$

is the number of positive divisors d of n.

We can see that $\rho(n)$ is negative if and only if n is a square number, since $\tau(n)$ is always odd when n is a perfect square (that is quite easy to prove).

OK, let us now have a look at a new function (I name it *P*, the capital Greek letter Rho):

$$P(n) = \sum_{k=1}^{n} \rho(k)$$

which can, of course, also be written as

$$P(n) = \sum_{k=1}^{n} (-k)^{\tau(k)}$$

The first values of this sequence is -1, 3, 12, -52, -27, 1269, 1318, … (recently approved as sequence A224914), and it continues with positive values until n=36, where *P*(36) suddenly becomes -100792120241072.
The sequence is then negative until we reach *P*(48) = 64840521809262990. After that, *P*(*n*) is always positive!

And here comes my new theorem (well, actually a conjecture):

Define *P*(*n*) as above. Then *P*(*n*) will not be negative for any *n* larger than 47.

To calculate the values in the *P*(*n*) sequence requires some programming skills. As an example, *P*(20000) has the value 21684898667257552647611856235715758072247691347157065921114617853497849841242870
2656076108344964687179907041068869527547983659163103580005653203079825147538557994876559326
7522654867541838735544979889617361898232220700243571313828111062277359888411262590613784776
66706702715483548559837701761253931028254903738773913458357516192256550376678878.

C++ source code

I used the following code in C++ for calculations (no optimized algorithm for finding the divisors, so that part would need to be updated to shorten the calculations). The implementation of the help class *Integer* is omitted. Contact me for further information.

```
#include <fstream>
#include <iomanip>
#include <utility>
#include <cassert>
#include <cmath>

using namespace std;
using namespace std::rel_ops;

/* Number of iterations */
#define SIZE 10000

/* Main console application */
int main() {
  /* Create file */
  ofstream file("data.txt");

  /* Number of positive divisors (tau) */
  int tau=0;

  /* Divisor products (pn), the sum (P) and the previous sum (P_prev) */
  Integer pn=1, P=0, P_prev=0;

  /* Max and min registrators */
  Integer largest=0, smallest=0;
  int n_largest=0, n_smallest=0;

  for(int n=1; n<=SIZE; pn=1,P_prev=P,n++) {
    /* Counter on screen */
    if(!(n%1000))
      cout<<n<<endl;

    /* Find divisors and calculate divisor product */
    tau=0;
    for(int d=1; d<=n; d++)
      if(!(n%d)) {
        tau++;
        pn=pn*d*(-d);
      }

    /* calculate the sum */
    P=P_prev+pn;
    /* Check for max and min */
    if(P>largest) {
      largest=P;
      n_largest=n;
    }
```

```
    if(P<smallest) {
        smallest=P;
        n_smallest=n;
      }
      /* Write result to file */
      file.ios::fixed;
      file<<setprecision(15);
      file<<"n = "<<n<<"\ttau("<<n<<") = "<<tau<<"\tpn("<<n<<") = "<<pn<<"\tP("<<n<<") = "<<P<<endl;
    }
    file<<endl<<endl<<"Smallest: P("<<n_smallest<<") = "<<smallest<<endl;
    file<<endl<<endl<<"Largest: P("<<n_largest<<") = "<<largest<<endl;
    file.close();
    return 0;
}
```

Can you prove it?

Proving the conjecture has shown to be harder than I first thought, and I need help from a mathematician. This is how I have been reasoning:

First, have a look at the following table. It contains the first 100 values of both n, $\tau(n)$, $\rho(n)$ and $P(n)$.

$n = 1$	**$\tau(1) = 1$**	**$\rho(1) = -1$**	**$P(1) = -1$**
$n = 2$	$\tau(2) = 2$	$\rho(2) = 4$	$P(2) = 3$
$n = 3$	$\tau(3) = 2$	$\rho(3) = 9$	$P(3) = 12$
$n = 4$	**$\tau(4) = 3$**	**$\rho(4) = -64$**	**$P(4) = -52$**
$n = 5$	$\tau(5) = 2$	$\rho(5) = 25$	**$P(5) = -27$**
$n = 6$	$\tau(6) = 4$	$\rho(6) = 1296$	$P(6) = 1269$
$n = 7$	$\tau(7) = 2$	$\rho(7) = 49$	$P(7) = 1318$
$n = 8$	$\tau(8) = 4$	$\rho(8) = 4096$	$P(8) = 5414$
$n = 9$	**$\tau(9) = 3$**	**$\rho(9) = -729$**	$P(9) = 4685$
$n = 10$	$\tau(10) = 4$	$\rho(10) = 10000$	$P(10) = 14685$
$n = 11$	$\tau(11) = 2$	$\rho(11) = 121$	$P(11) = 14806$
$n = 12$	$\tau(12) = 6$	$\rho(12) = 2985984$	$P(12) = 3000790$
$n = 13$	$\tau(13) = 2$	$\rho(13) = 169$	$P(13) = 3000959$
$n = 14$	$\tau(14) = 4$	$\rho(14) = 38416$	$P(14) = 3039375$
$n = 15$	$\tau(15) = 4$	$\rho(15) = 50625$	$P(15) = 3090000$
$n = 16$	**$\tau(16) = 5$**	**$\rho(16) = -1048576$**	$P(16) = 2041424$
$n = 17$	$\tau(17) = 2$	$\rho(17) = 289$	$P(17) = 2041713$
$n = 18$	$\tau(18) = 6$	$\rho(18) = 34012224$	$P(18) = 36053937$
$n = 19$	$\tau(19) = 2$	$\rho(19) = 361$	$P(19) = 36054298$
$n = 20$	$\tau(20) = 6$	$\rho(20) = 64000000$	$P(20) = 100054298$
$n = 21$	$\tau(21) = 4$	$\rho(21) = 194481$	$P(21) = 100248779$
$n = 22$	$\tau(22) = 4$	$\rho(22) = 234256$	$P(22) = 100483035$
$n = 23$	$\tau(23) = 2$	$\rho(23) = 529$	$P(23) = 100483564$
$n = 24$	$\tau(24) = 8$	$\rho(24) = 110075314176$	$P(24) = 110175797740$
$n = 25$	**$\tau(25) = 3$**	**$\rho(25) = -15625$**	$P(25) = 110175782115$
$n = 26$	$\tau(26) = 4$	$\rho(26) = 456976$	$P(26) = 110176239091$
$n = 27$	$\tau(27) = 4$	$\rho(27) = 531441$	$P(27) = 110176770532$
$n = 28$	$\tau(28) = 6$	$\rho(28) = 481890304$	$P(28) = 110658660836$
$n = 29$	$\tau(29) = 2$	$\rho(29) = 841$	$P(29) = 110658661677$
$n = 30$	$\tau(30) = 8$	$\rho(30) = 656100000000$	$P(30) = 766758661677$
$n = 31$	$\tau(31) = 2$	$\rho(31) = 961$	$P(31) = 766758662638$
$n = 32$	$\tau(32) = 6$	$\rho(32) = 1073741824$	$P(32) = 767832404462$
$n = 33$	$\tau(33) = 4$	$\rho(33) = 1185921$	$P(33) = 767833590383$
$n = 34$	$\tau(34) = 4$	$\rho(34) = 1336336$	$P(34) = 767834926719$
$n = 35$	$\tau(35) = 4$	$\rho(35) = 1500625$	$P(35) = 767836427344$
$n = 36$	**$\tau(36) = 9$**	**$\rho(36) = -101559956668416$**	**$P(36) = -100792120241072$**
$n = 37$	$\tau(37) = 2$	$\rho(37) = 1369$	**$P(37) = -100792120239703$**
$n = 38$	$\tau(38) = 4$	$\rho(38) = 2085136$	**$P(38) = -100792118154567$**
$n = 39$	$\tau(39) = 4$	$\rho(39) = 2313441$	**$P(39) = -100792115841126$**
$n = 40$	$\tau(40) = 8$	$\rho(40) = 6553600000000$	**$P(40) = -94238515841126$**
$n = 41$	$\tau(41) = 2$	$\rho(41) = 1681$	**$P(41) = -94238515839445$**
$n = 42$	$\tau(42) = 8$	$\rho(42) = 9682651996416$	**$P(42) = -84555863843029$**
$n = 43$	$\tau(43) = 2$	$\rho(43) = 1849$	**$P(43) = -84555863841180$**
$n = 44$	$\tau(44) = 6$	$\rho(44) = 7256313856$	**$P(44) = -84548607527324$**
$n = 45$	$\tau(45) = 6$	$\rho(45) = 8303765625$	**$P(45) = -84540303761699$**
$n = 46$	$\tau(46) = 4$	$\rho(46) = 4477456$	**$P(46) = -84540299284243$**
$n = 47$	$\tau(47) = 2$	$\rho(47) = 2209$	**$P(47) = -84540299282034$**
$n = 48$	$\tau(48) = 10$	$\rho(48) = 64925062108545024$	$P(48) = 64840521809262990$
$n = 49$	**$\tau(49) = 3$**	**$\rho(49) = -117649$**	$P(49) = 64840521809145341$
$n = 50$	$\tau(50) = 6$	$\rho(50) = 15625000000$	$P(50) = 64840537434145341$
$n = 51$	$\tau(51) = 4$	$\rho(51) = 6765201$	$P(51) = 64840537440910542$
$n = 52$	$\tau(52) = 6$	$\rho(52) = 19770609664$	$P(52) = 64840557211520206$

$n = 53$	$\tau(53) = 2$	$\rho(53) = 2809$	$P(53) = 64840557211523015$
$n = 54$	$\tau(54) = 8$	$\rho(54) = 72301961339136$	$P(54) = 64912859172862151$
$n = 55$	$\tau(55) = 4$	$\rho(55) = 9150625$	$P(55) = 64912859182012776$
$n = 56$	$\tau(56) = 8$	$\rho(56) = 96717311574016$	$P(56) = 65009576493586792$
$n = 57$	$\tau(57) = 4$	$\rho(57) = 10556001$	$P(57) = 65009576504142793$
$n = 58$	$\tau(58) = 4$	$\rho(58) = 11316496$	$P(58) = 65009576515459289$
$n = 59$	$\tau(59) = 2$	$\rho(59) = 3481$	$P(59) = 65009576515462770$
$n = 60$	$\tau(60) = 12$	$\rho(60) = 2176782336000000000000$	$P(60) = 2176847345576515462770$
$n = 61$	$\tau(61) = 2$	$\rho(61) = 3721$	$P(61) = 2176847345576515466491$
$n = 62$	$\tau(62) = 4$	$\rho(62) = 14776336$	$P(62) = 2176847345576530242827$
$n = 63$	$\tau(63) = 6$	$\rho(63) = 62523502209$	$P(63) = 2176847345639053745036$
$\boldsymbol{n = 64}$	$\boldsymbol{\tau(64) = 7}$	$\boldsymbol{\rho(64) = -4398046511104}$	$P(64) = 2176847341241007233932$
$n = 65$	$\tau(65) = 4$	$\rho(65) = 17850625$	$P(65) = 2176847341241025084557$
$n = 66$	$\tau(66) = 8$	$\rho(66) = 360040606269696$	$P(66) = 2176847701281631354253$
$n = 67$	$\tau(67) = 2$	$\rho(67) = 4489$	$P(67) = 2176847701281631358742$
$n = 68$	$\tau(68) = 6$	$\rho(68) = 98867482624$	$P(68) = 2176847701380498841366$
$n = 69$	$\tau(69) = 4$	$\rho(69) = 22667121$	$P(69) = 2176847701380521508487$
$n = 70$	$\tau(70) = 8$	$\rho(70) = 576480100000000$	$P(70) = 2176848277860621508487$
$n = 71$	$\tau(71) = 2$	$\rho(71) = 5041$	$P(71) = 2176848277860621513528$
$n = 72$	$\tau(72) = 12$	$\rho(72) = 19408409961765342806016$	$P(72) = 21585258239625964319544$
$n = 73$	$\tau(73) = 2$	$\rho(73) = 5329$	$P(73) = 21585258239625964324873$
$n = 74$	$\tau(74) = 4$	$\rho(74) = 29986576$	$P(74) = 21585258239625994311449$
$n = 75$	$\tau(75) = 6$	$\rho(75) = 177978515625$	$P(75) = 21585258239803972827074$
$n = 76$	$\tau(76) = 6$	$\rho(76) = 192699928576$	$P(76) = 21585258239996672755650$
$n = 77$	$\tau(77) = 4$	$\rho(77) = 35153041$	$P(77) = 21585258239996707908691$
$n = 78$	$\tau(78) = 8$	$\rho(78) = 1370114370683136$	$P(78) = 21585259610111078591827$
$n = 79$	$\tau(79) = 2$	$\rho(79) = 6241$	$P(79) = 21585259610111078598068$
$n = 80$	$\tau(80) = 10$	$\rho(80) = 10737418240000000000$	$P(80) = 21595997028351078598068$
$\boldsymbol{n = 81}$	$\boldsymbol{\tau(81) = 5}$	$\boldsymbol{\rho(81) = -3486784401}$	$P(81) = 21595997028347591813667$
$n = 82$	$\tau(82) = 4$	$\rho(82) = 45212176$	$P(82) = 21595997028347637025843$
$n = 83$	$\tau(83) = 2$	$\rho(83) = 6889$	$P(83) = 21595997028347637032732$
$n = 84$	$\tau(84) = 12$	$\rho(84) = 123410307017276135571456$	$P(84) = 145006304045623772604188$
$n = 85$	$\tau(85) = 4$	$\rho(85) = 52200625$	$P(85) = 145006304045623824804813$
$n = 86$	$\tau(86) = 4$	$\rho(86) = 54700816$	$P(86) = 145006304045623879505629$
$n = 87$	$\tau(87) = 4$	$\rho(87) = 57289761$	$P(87) = 145006304045623936795390$
$n = 88$	$\tau(88) = 8$	$\rho(88) = 3596345248055296$	$P(88) = 145006307641969184850686$
$n = 89$	$\tau(89) = 2$	$\rho(89) = 7921$	$P(89) = 145006307641969184858607$
$n = 90$	$\tau(90) = 12$	$\rho(90) = 282429536481000000000000$	$P(90) = 427435844122969184858607$
$n = 91$	$\tau(91) = 4$	$\rho(91) = 68574961$	$P(91) = 427435844122969253433568$
$n = 92$	$\tau(92) = 6$	$\rho(92) = 606355001344$	$P(92) = 427435844123575608434912$
$n = 93$	$\tau(93) = 4$	$\rho(93) = 74805201$	$P(93) = 427435844123575683240113$
$n = 94$	$\tau(94) = 4$	$\rho(94) = 78074896$	$P(94) = 427435844123575761315009$
$n = 95$	$\tau(95) = 4$	$\rho(95) = 81450625$	$P(95) = 427435844123575842765634$
$n = 96$	$\tau(96) = 12$	$\rho(96) = 612709757329767363772416$	$P(96) = 1040145601453343206538050$
$n = 97$	$\tau(97) = 2$	$\rho(97) = 9409$	$P(97) = 1040145601453343206547459$
$n = 98$	$\tau(98) = 6$	$\rho(98) = 885842380864$	$P(98) = 1040145601454229048928323$
$n = 99$	$\tau(99) = 6$	$\rho(99) = 941480149401$	$P(99) = 1040145601455170529077724$
$\boldsymbol{n = 100}$	$\boldsymbol{\tau(100) = 9}$	$\boldsymbol{\rho(100) = -1000000000000000000}$	$P(100) = 1040144601455170529077724$

As n increases, the distance between the negative terms in the sum $P(n)$ also increases, since a term, i.e. $\rho(n)$, can only be negative when n is a *square number* (and the distance between square numbers increases). So, even though some of the negative terms are "very negative", such as $\rho(14400)$= -94806257274301054768140939840951508420 10498179757960567157629413745492904074056364495045084639345243622288250814421184698916540916 95718400000000000000000000000000000000000000000000000000000000000000000000000000000000000000 0000000000000000000000000000000000000000, they will never outweigh the positive terms.

Furthermore, we know that highly composite numbers are never square (except for n=1, n=4 and n=36). This strongly indicates that the conjecture above is true, but it does not prove it.

I have been looking for various recognizably structures in both the $\rho(n)$ sequence and the $P(n)$ sequence, but without getting anywhere. So, dear math student, be the first to prove it!

Simon Jensen, Gothenburg, April 2013

Contact information

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